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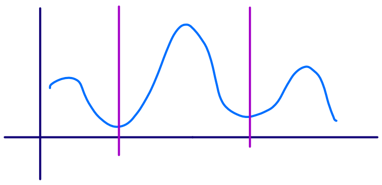
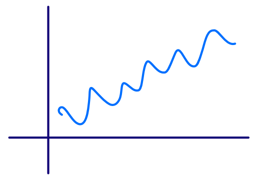
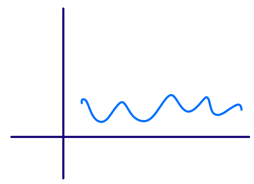
Module 4 - Spectral Representation

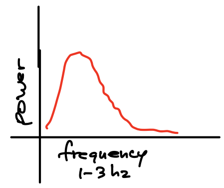
$\gamma(h)$





VS.





Periodogram → is there a cycle happening every day? Or is there some slower thing happening?

$\sin(x + 2\pi) = \sin(x)$

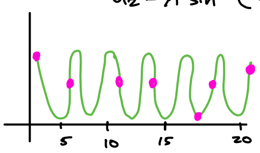
→ once you go past 2π, goes back to

$$X_t = A\cos(2\pi\omega t + \phi)$$

A → how big it is (weight); 2π → constant; ω → freq.; t → time; ϕ → phase

$$X_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t)$$

$$U_1 = A\cos(\phi)$$
$$U_2 = A\sin(\phi)$$



fast moving wave. From the point of view of U, trying to fit the data, can't tell the difference → aliasing (quiz question)

Same periodic functions apply to images → Where you can't tell the difference btwn high frequency stuff & low freq. stuff

folding frequency

$$X_t = \sum_{k=1}^{q} U_{k1}\cos(2\pi\omega_k t) + U_k\sin(2\pi\omega_k t) \rightarrow$$ signal as a sum of frequencies

$\gamma(h) = \mathrm{cov}(x_{t+h}, x_t)$; must be stationary; hence this is defined

$$\gamma(h) = \sum_{k=1}^{q} \sigma^2\cos(2\pi\omega_k h)$$

σ^2 → variance; 2π → const; ω_k → freq. of sin or cos; h → lag

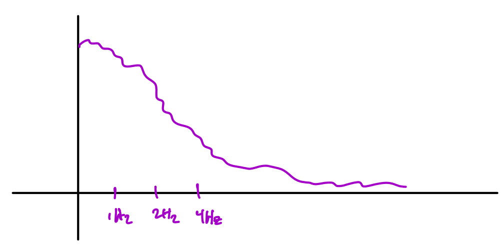
$\therefore$

$$\gamma(0) = \mathrm{var}(x_t) = \sum_{k=1}^{q} \sigma_k^2$$

$$X_t = a_0 + \sum_{j=1}^{(n-1)/2} a_j\cos\left(2\pi t \tfrac{j}{n}\right) + b_j\sin\left(2\pi t \tfrac{j}{n}\right)$$

$$P_{j/n} = a_j^2 + b_j^2$$

→ periodogram vector



Spectral Density

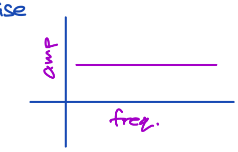
$$\gamma(h) = \mathrm{cov}(x_{t+h}, x_t)$$

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}$$

ω → freq.

White noise

flat (= amount @ every freq.)



Linear Filters

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}$$

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty$$

$$A_{yx}(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i \omega j}$$

i → = $\sqrt{-1}$ (imaginary #)

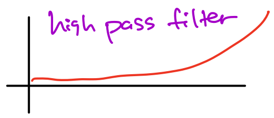
e → exponential → 2.78

$$f_Y(\omega) = |A_{yx}(\omega)|^2 \, f_x(\omega)$$ → convolution

$f_x(\omega)$ → spectral density of our input

$A_{yx}(\omega)$ → spectral density that we calculate from the filter

$$y_t = \nabla x_t = x_t - x_{t-1}$$
↳ differencing



Discrete Fourier Transform → used to calculate periodogram & estimate spectral density

$$d(\omega_j) = n^{-1/2} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t}$$

$j = 0, \dots n-1$

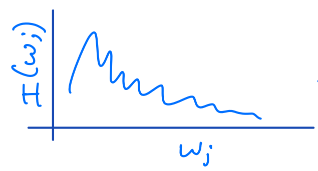
$\omega_j = j/n$

$$|d(\omega_j)^2| = I(\omega_j)$$ periodogram

$$x_t = n^{-1/2} \sum_{j=0}^{n-1} d(\omega_j) e^{2\pi i \omega_j t}$$

$$e^{ix} = \cos x + i \sin x$$

$I(0) = n\bar{x}^2$



→ read-out of how much frequency is present in ea. signal

$$d_c(\omega_j) = n^{-1/2} \sum_{t=1}^{n} x_j \cos(2\pi \omega_j t) \; ; \; d_c(\omega_j) \sim N(0, \tfrac{1}{2} f(\omega_j))$$

$d_c(\omega_j)$ → right out of DCF

$f(\omega_j)$ → theoretical quantity from spectral density

it's basically going to have a normal distribution.

$$d_c(\omega_j) \perp d_s(\omega_j)$$

→ independent; and it's also independent of the other frequencies b/c orthogonal

$$E[I(\omega_j)] \approx f(\omega_j)$$

$f(\omega_j)$ → population version (what you wanted to estimate) quantity

$E[I(\omega_j)]$ → sample version (what you got from data/algorithm/whole procedure)

$$\text{var } I(\omega_j) = f^2 \omega(j)$$ → not zero → constant (doesn't depend on size of dataset)

bandwidth

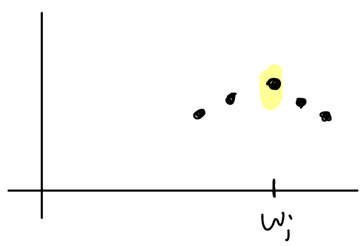
$$B = \left\{ \omega_j + \tfrac{k}{n} : k = 0, \pm 1, m \right\}$$

$\{$ → set

$:$ → such that

$L = 2m+1$

$$\bar{f}(\omega_j) = \frac{1}{L} \sum_{k=-m}^{m} I(\omega_j + \tfrac{k}{n})$$



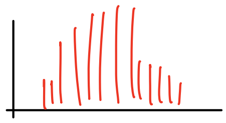
$$E[\bar{f}(\omega_j)] = f(\omega_j)$$ → unbiased

$$\text{var}[\bar{f}(\omega_j)] = \frac{f^2(\omega_j)}{L}$$

→ less variance than before; improved our estimate

$$\bar{f}(\omega_j) = \sum_{k=-m}^{m} h_k I(\omega_j + \tfrac{k}{n})$$

h_k → weights → $\sum_{k=-m}^{m} h_k = 1$



$$\frac{f^2(\omega_j)}{L_h}$$